

Smoothing metrics on closed Riemannian manifolds through the Ricci flow

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Abstract

Under the assumption of the uniform local Sobolev inequality, it is proved that Riemannian metrics with an absolute Ricci curvature bound and a small Riemannian curvature integral bound can be smoothed to having a sectional curvature bound. This partly extends previous a priori estimates of Ye Li (J. Geom. Anal. 17 (2007) 495-511; Advances in Mathematics 223 (2010) 1924-1957).

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1. Introduction

If a Riemannian manifold has bounded sectional curvature, then its geometric structure is better understood than that with weaker curvature bounds, say Ricci curvature bounds. Thus it is of significance to deform or smooth a Riemannian metric with a Ricci curvature bound to a metric with a sectional curvature bound. One way to do this is using the Ricci flow. In this regard we refer the reader to the pioneer works [3, 6, 17, 18]. If the initial metric has bounded curvatures, one can show the short time existence of the Ricci flow and obtain the covariant derivatives bounds for the curvature tensors along the Ricci flow [2, 14]. If the initial metric has bounded Ricci curvature, under some additional assumption on conjugate radius, Dai, etc. studied how to deform the metric on closed manifolds [6]. Also one can deform a metric locally by using the local Ricci flow [11, 12, 13, 16, 18]. Throughout this paper, we use $\text{Rm}(g)$ and $\text{Ric}(g)$ to denote the Riemannian curvature tensor and Ricci tensor with respect to the metric g respectively. Our main result is the following:

Theorem 1.1. *Assume (M, g_0) is a closed Riemannian manifold of dimension n ($n \geq 3$) and $|\text{Ric}(g_0)| \leq K$ for some constant K . Let $B_r(x)$ be a geodesic ball centered at $x \in M$ with radius r . Suppose there exists a constant $A_0 > 0$ such that for all $x \in M$ and some $r \leq \min(\frac{1}{2}\text{diam}(g_0), 1)$*

$$\left(\int_{B_r(x)} |u|^{\frac{2n}{n-2}} dv_{g_0} \right)^{(n-2)/n} \leq A_0 \int_{B_r(x)} |\nabla_{g_0} u|^2 dv_{g_0}, \quad \forall u \in C_0^\infty(B_r(x)). \quad (1.1)$$

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Then there exist constants ϵ, c_1, c_2 depending only on n and K such that if

$$\left(\int_{B_r(x)} |\text{Rm}(g_0)|^{\frac{n}{2}} dv_{g_0} \right)^{2/n} \leq \epsilon A_0^{-1} \quad \text{for all } x \in M, \quad (1.2)$$

then the Ricci flow

$$\begin{cases} \frac{\partial g}{\partial t} = -2\text{Ric}(g), \\ g(0) = g_0 \end{cases} \quad (1.3)$$

has a unique smooth solution satisfying the following estimates

$$|g(t) - g_0|_{g_0} \leq c_2 t^{\frac{2}{n+2}}, \quad (1.4)$$

$$|\text{Rm}(g(t))|_{\infty} \leq c_2 t^{-1}, \quad (1.5)$$

$$|\text{Ric}(g(t))|_{\infty} \leq c_2 t^{-\frac{n}{n+2}} \quad (1.6)$$

for $0 \leq t \leq T$ with $T \geq c_1 \min(r^2, K^{-1})$.

When (M, g_0) is a complete noncompact Riemannian manifold, similar results were obtained by Ye Li [13] and G. Xu [16]. The assumptions of [13] is much weaker than (1.1) and (1.2) in case $n = 4$. It comes from Cheeger and Tian's work [5] concerning the collapsing Einstein 4-manifolds. Here Theorem 1.1 is just the beginning of extending the results [5, 12, 13], which may depend on the Gauss-Bonnet-Chern formula, to general dimensional case.

For the proof of Theorem 1.1, we follow the lines of [6, 7, 13, 18]. Let's roughly describe the idea. First it is well known [10, 8] that the Ricci flow (1.3) has a unique smooth solution $g(t)$ for a very short time interval. Using Moser's iteration and Gromov's covering argument, we derive a priori estimates on $\text{Rm}(g(t))$ and $\text{Ric}(g(t))$. Let $[0, T_{\max}]$ be a maximum time interval on which $g(t)$ exists. Then based on those a priori estimates, T_{\max} has the desired lower bound.

Such kind of results are very useful when considering the relation between curvature and topology [1, 6, 12]. Using Theorem 1.1, we can easily generalize Gromov's almost flat manifold theorem [9]. Particularly one has the following:

Theorem 1.2. *There exist constants ϵ and δ depending only on n and K such that if a closed Riemannian manifold (M, g_0) satisfies $|\text{Ric}(g_0)| \leq K$, $\text{diam}(g_0) \leq \delta$, (1.1) and (1.2) hold for all $x \in M$, then the universal covering space of (M, g_0) is \mathbb{R}^n . If all the above hypothesis on (M, g_0) are satisfied and moreover the fundamental group $\pi(g_0)$ is commutative, then (M, g_0) is diffeomorphic to a torus.*

Before ending this introduction, we would like to mention [15] for local regularity estimates for Riemannian curvatures. The remaining part of the paper is organized as follows. In Sect. 2, we derive two weak maximum principles by using the Moser's iteration. In Sect. 3, we estimate the time interval on which the solution of Ricci flow exists, and prove Theorem 1.1. Finally Theorem 1.2 is proved in Sect. 4.

2. Weak maximum principles

In this section, following the lines of [13, 18], we give two maximum principles via the Moser's iteration. Throughout this section the manifolds need not to be compact. Suppose

$(M, g(t))$ are complete Riemannian manifolds for $0 \leq t \leq T$. Let $\nabla_{g(t)}$ denote the covariant differentiation with respect to $g(t)$ and $-\Delta_{g(t)}$ be the corresponding Laplace-Beltrami operator, which will be also denoted by ∇ and $-\Delta$ for simplicity, the reader can easily recognize it from the context. Let A be a constant such that for all $t \in [0, T]$,

$$\left(\int_{B_r(x)} |u|^{\frac{2n}{n-2}} dv_t \right)^{(n-2)/n} \leq A \int_{B_r(x)} |\nabla u|^2 dv_t, \quad \forall u \in C_0^\infty(B_r(x)), \quad (2.1)$$

where $dv_t = dv_{g(t)}$. Assume that for all $0 \leq t \leq T$,

$$\frac{1}{2}g_0 \leq g(t) \leq 2g_0 \quad \text{on } M. \quad (2.2)$$

Here and in the sequel, all geodesic balls are defined with respect to g_0 .

Firstly we have the following maximum principle:

Theorem 2.1. *Let $(M, g(t))$ be complete Riemannian manifolds and (2.1), (2.2) are satisfied for $0 \leq t \leq T$. Let $f(x, t)$ be such that*

$$\frac{\partial f}{\partial t} \leq \Delta f + uf \quad \text{on } B_r(x) \times [0, T] \quad (2.3)$$

with $f \geq 0$, $u \geq 0$,

$$\frac{\partial}{\partial t} dv_t \leq cudv_t, \quad (2.4)$$

for some constant c depending only on n and for some $q > n$

$$\left(\int_{B_r(x)} u^{\frac{q}{2}} dv_t \right)^{\frac{2}{q}} \leq \mu t^{-\frac{q-n}{q}}, \quad (2.5)$$

where $\mu > 0$ is a constant. Then for any $p > 1$, $t \in [0, T]$, we have

$$f(x, t) \leq CA^{\frac{n}{2p}} \left(\frac{1 + A^{\frac{n}{q-n}} \mu^{\frac{q}{q-n}}}{t} + \frac{1}{r^2} \right)^{\frac{n+2}{2p}} \left(\int_0^T \int_{B_r(x)} f^p dv_t \right)^{\frac{1}{p}}, \quad (2.6)$$

where C is a constant depending only on n , q and p .

Proof. Let η be a nonnegative Lipschitz function supported in $B_r(x)$. We first consider the case $p \geq 2$. By the partial differential inequality (2.3) and (2.4), we have

$$\frac{1}{p} \frac{\partial}{\partial t} \int \eta^2 f^p dv_t \leq \int \eta^2 f^{p-1} \Delta f dv_t + C_1 \int u f^p \eta^2 dv_t,$$

where C_1 is a constant depending only on n . Integration by parts implies

$$\begin{aligned}
\int \eta^2 f^{p-1} \Delta f dv_t &= -2 \int \eta f^{p-1} \nabla \eta \nabla f dv_t - (p-1) \int \eta^2 f^{p-2} |\nabla f|^2 dv_t \\
&= -\frac{4}{p} \int \left(f^{\frac{p}{2}} \nabla \eta \nabla (\eta f^{\frac{p}{2}}) - |\nabla \eta|^2 f^p \right) dv_t - \frac{4(p-1)}{p^2} \\
&\quad \times \int \left(|\nabla (\eta f^{\frac{p}{2}})|^2 + |\nabla \eta|^2 f^p - 2 f^{\frac{p}{2}} \nabla \eta \nabla (\eta f^{\frac{p}{2}}) \right) dv_t \\
&= -\frac{4(p-1)}{p^2} \int |\nabla (\eta f^{\frac{p}{2}})|^2 dv_t + \frac{4}{p^2} \int |\nabla \eta|^2 f^p dv_t \\
&\quad + \frac{4p-8}{p^2} \int f^{\frac{p}{2}} \nabla \eta \nabla (\eta f^{\frac{p}{2}}) dv_t \\
&\leq -\frac{2}{p} \int |\nabla (\eta f^{\frac{p}{2}})|^2 dv_t + \frac{2}{p} \int |\nabla \eta|^2 f^p dv_t.
\end{aligned}$$

Here we have used the elementary inequality $2ab \leq a^2 + b^2$. By the Hölder inequality, we have

$$\int u f^p \eta^2 dv_t \leq \left(\int u^{\frac{q}{2}} dv_t \right)^{\frac{2}{q}} \left(\int (\eta^2 f^p)^{\alpha q_1} dv_t \right)^{\frac{1}{q_1}} \left(\int (\eta^2 f^p)^{(1-\alpha)q_2} dv_t \right)^{\frac{1}{q_2}},$$

where $\frac{1}{q_1} + \frac{1}{q_2} + \frac{2}{q} = 1$ and $0 < \alpha < 1$. Let $\alpha q_1 = \frac{n}{n-2}$ and $(1-\alpha)q_2 = 1$. This implies $q_1 = \frac{q}{n-2}$, $q_2 = \frac{q}{q-n}$ and $\alpha = \frac{n}{q}$. Using the Sobolev inequality (2.1) and the Young inequality, we obtain

$$\begin{aligned}
\int u f^p \eta^2 dv_t &\leq \mu t^{-\frac{q-n}{q}} \left(\int (\eta^2 f^p)^{\frac{n}{n-2}} dv_t \right)^{\frac{n-2}{q}} \left(\int \eta^2 f^p dv_t \right)^{\frac{q-n}{q}} \\
&\leq \mu t^{-\frac{q-n}{q}} \left(A \int |\nabla (\eta f^{\frac{p}{2}})|^2 dv_t \right)^{\frac{n}{q}} \left(\int \eta^2 f^p dv_t \right)^{\frac{q-n}{q}} \\
&\leq \frac{1}{p C_1} \int |\nabla (\eta f^{\frac{p}{2}})|^2 dv_t + C_2 p^{\frac{n}{q-n}} \mu^{\frac{q}{q-n}} A^{\frac{n}{q-n}} t^{-1} \int \eta^2 f^p dv_t
\end{aligned}$$

for some constant C_2 depending only on n and q . Combining all the above estimates one has

$$\begin{aligned}
\frac{\partial}{\partial t} \int \eta^2 f^p dv_t + \int |\nabla (\eta f^{\frac{p}{2}})|^2 dv_t &\leq 2 \int |\nabla \eta|^2 f^p dv_t \\
&\quad + C_1 C_2 p^{\frac{q}{q-n}} \mu^{\frac{q}{q-n}} A^{\frac{n}{q-n}} t^{-1} \int \eta^2 f^p dv_t.
\end{aligned} \tag{2.7}$$

For $0 < \tau < \tau' < T$, let

$$\psi(t) = \begin{cases} 0, & 0 \leq t \leq \tau \\ \frac{t-\tau}{\tau'-\tau}, & \tau \leq t \leq \tau' \\ 1, & \tau' \leq t \leq T. \end{cases}$$

Multiplying (2.7) by ψ , we have

$$\begin{aligned}
\frac{\partial}{\partial t} \left(\psi \int \eta^2 f^p dv_t \right) + \psi \int |\nabla (\eta f^{\frac{p}{2}})|^2 dv_t &\leq 2\psi \int |\nabla \eta|^2 f^p dv_t \\
&\quad + \left(C_1 C_2 p^{\frac{q}{q-n}} \mu^{\frac{q}{q-n}} A^{\frac{n}{q-n}} t^{-1} \psi + \psi' \right) \int \eta^2 f^p dv_t.
\end{aligned} \tag{2.8}$$

Assume $\tau < \tau' < t \leq T$. Since on the time interval $[\tau, \tau']$

$$0 \leq \frac{\psi(t)}{t} = \frac{1}{\tau' - \tau} - \frac{\tau}{\tau' - \tau} \frac{1}{t} \leq \frac{1}{\tau' - \tau} \left(1 - \frac{\tau}{\tau'}\right) = \frac{1}{\tau'},$$

and on the time interval $[\tau', T]$

$$\frac{1}{T} \leq \frac{\psi(t)}{t} \leq \frac{1}{\tau'},$$

we have

$$\int_{\tau}^t \frac{\psi(t)}{t} \left(\int \eta^2 f^p dv_t \right) dt \leq \frac{1}{\tau'} \int_{\tau}^t \int \eta^2 f^p dv_t dt. \quad (2.9)$$

Notice that $0 \leq \psi \leq 1$ and $0 \leq \psi' \leq \frac{1}{\tau' - \tau}$. Integrating the differential inequality (2.8) from τ to t , we obtain by using (2.9)

$$\begin{aligned} \int \eta^2 f^p dv_t + \int_{\tau'}^t \int |\nabla(\eta f^{\frac{p}{2}})|^2 dv_t dt &\leq 2 \int_{\tau}^t \int |\nabla \eta|^2 f^p dv_t dt \\ &+ \left(\frac{C_1 C_2 p^{\frac{q}{q-n}} \mu^{\frac{q}{q-n}} A^{\frac{n}{q-n}}}{\tau'} + \frac{1}{\tau' - \tau} \right) \int_{\tau}^T \int \eta^2 f^p dv_t dt. \end{aligned}$$

Applying this estimate and the Sobolev inequality we derive

$$\begin{aligned} \int_{\tau'}^T \int f^{p(1+\frac{2}{n})} \eta^{2+\frac{1}{n}} dv_t dt &\leq \int_{\tau'}^T \left(\int \eta^2 f^p dv_t \right)^{\frac{2}{n}} \left(\int f^{\frac{pm}{n-2}} \eta^{\frac{2n}{n-2}} dv_t \right)^{\frac{n-2}{n}} dt \\ &\leq A \left(\sup_{\tau' \leq t \leq T} \int \eta^2 f^p \right)^{\frac{2}{n}} \int_{\tau'}^T \int |\nabla(\eta f^{\frac{p}{2}})|^2 dv_t dt \\ &\leq A \left[2 \int_{\tau}^t \int |\nabla \eta|^2 f^p dv_t dt + \left(\frac{C_1 C_2 p^{\frac{q}{q-n}} \mu^{\frac{q}{q-n}} A^{\frac{n}{q-n}}}{\tau'} \right. \right. \\ &\quad \left. \left. + \frac{1}{\tau' - \tau} \right) \int_{\tau}^T \int \eta^2 f^p dv_t dt \right]^{1+\frac{2}{n}}. \end{aligned} \quad (2.10)$$

For $p \geq p_0 \geq 2$ and $0 \leq \tau \leq T$, we set

$$H(p, \tau, r) = \int_{\tau}^T \int_{B_r(x)} f^p dv_t dt,$$

where $B_r(x)$ is the geodesic ball centered at x with radius r measured in $g(0)$. Choosing a suitable cut-off function η and noting that $|\nabla \eta|_t \leq 2|\nabla \eta|_0$, we obtain from (2.10)

$$\begin{aligned} &H\left(p\left(1 + \frac{2}{n}\right), \tau', r\right) \\ &\leq AC_3 \left(\frac{p^{\frac{q}{q-n}} \mu^{\frac{q}{q-n}} A^{\frac{n}{q-n}}}{\tau'} + \frac{1}{\tau' - \tau} + \frac{1}{(r' - r)^2} \right)^{1+\frac{2}{n}} H(p, \tau, r')^{1+\frac{2}{n}}, \end{aligned} \quad (2.11)$$

where $0 < r < r'$, C_3 is a constant depending only on n and q . Set

$$v = 1 + \frac{2}{n}, \quad p_k = p_0 v^k, \quad \tau_k = (1 - v^{-\frac{qk}{q-n}})t, \quad r_k = (1 + v^{-\frac{qk}{q-n}})r/2.$$

Then the inequality (2.11) gives

$$H(p_{k+1}, \tau_{k+1}, r_{k+1}) \leq AC_3 \left(\frac{1 + p_0^{\frac{q}{q-n}} \mu^{\frac{q}{q-n}} A^{\frac{n}{q-n}}}{t} + \frac{1}{r^2} \right)^\eta \eta^{kv} H(p_k, \tau_k, r_k)^v,$$

where $\eta = \nu^{\frac{2q}{q-n}}$. It follows that

$$\begin{aligned} & H(p_{k+1}, \tau_{k+1}, r_{k+1})^{\frac{1}{p_{k+1}}} \\ & \leq (AC_3)^{\frac{1}{p_{k+1}}} \left(\frac{1 + p_0^{\frac{q}{q-n}} \mu^{\frac{q}{q-n}} A^{\frac{n}{q-n}}}{t} + \frac{1}{r^2} \right)^{\frac{1}{p_k}} \eta^{\frac{k}{p_k}} H(p_k, \tau_k, r_k)^{\frac{1}{p_k}}. \end{aligned}$$

Hence we obtain for any fixed k

$$\begin{aligned} H(p_{k+1}, \tau_{k+1}, r_{k+1})^{\frac{1}{p_{k+1}}} & \leq (AC_3)^{\sum_{j=0}^k \frac{1}{p_{j+1}}} \left(\frac{1 + p_0^{\frac{q}{q-n}} \mu^{\frac{q}{q-n}} A^{\frac{n}{q-n}}}{t} + \frac{1}{r^2} \right)^{\sum_{j=0}^k \frac{1}{p_j}} \\ & \quad \eta^{\sum_{j=0}^k \frac{j}{p_j}} H(p_0, \tau_0, r_0)^{\frac{1}{p_0}}. \end{aligned}$$

Passing to the limit $k \rightarrow \infty$, one concludes

$$f(x, t) \leq (CA)^{\frac{n}{2p_0}} \left(\frac{1 + (p_0 \mu)^{\frac{q}{q-n}} A^{\frac{n}{q-n}}}{t} + \frac{1}{r^2} \right)^{\frac{n+2}{2p_0}} \left(\int_0^T \int f^{p_0} dv_t dt \right)^{\frac{1}{p_0}}.$$

This proves (2.6) in the case $p \geq 2$.

Assuming f satisfies (2.3) and $f \geq 0$. We define a sequence of functions

$$f_j = f + 1/j, \quad j \in \mathbb{N}.$$

Then f_j also satisfies (2.3) and $f_j^{p/2}$ is Lipschitz continuous for $1 < p < 2$. The same argument as the case $p \geq 2$ also yields

$$f_j(x, t) \leq (CA)^{\frac{n}{2p_0}} \left(\frac{1 + (p_0 \mu)^{\frac{q}{q-n}} A^{\frac{n}{q-n}}}{t} + \frac{1}{r^2} \right)^{\frac{n+2}{2p_0}} \left(\int_0^T \int f_j^{p_0} dv_t dt \right)^{\frac{1}{p_0}}$$

for some constant C depending only on n and q , where $1 < p_0 < 2$. Passing to the limit $j \rightarrow \infty$, we can see that (2.6) holds when $1 < p < 2$. \square

To proceed we need the following covering lemma belonging to M. Gromov.

Lemma 2.2 ([4], Proposition 3.11). *Let (M, g) be a complete Riemannian manifold, the Ricci curvature of M satisfy $\text{Ric}(g) \geq (n-1)H$. Then given $r, \epsilon > 0$ and $p \in M$, there exists a covering, $B_r(p) \subset \cup_{i=1}^N B_\epsilon(p_i)$, (p_i in $B_r(p)$) with $N \leq N_1(n, Hr^2, r/\epsilon)$. Moreover, the multiplicity of this covering is at most $N_2(n, Hr^2)$.*

For any complete Riemannian manifold (M, g_0) of dimension n with $|\text{Ric}(g_0)| \leq K$, it follows from Lemma 2.2 that there exists an absolute constant N depending only on K and n such that

$$B_{2r}(x) \subset \cup_{i=1}^N B_r(y_i), \quad y_i \in B_{\frac{1}{2}r}(x). \quad (2.12)$$

Suppose (2.1) and (2.2) hold for all $x \in M$ and $0 \leq t \leq T$, $g(0) = g_0$. Let $f(x, t)$ and $u(x, t)$ be two nonnegative functions satisfying

$$\frac{\partial f}{\partial t} \leq \Delta f + C_0 f^2, \quad \frac{\partial u}{\partial t} \leq \Delta u + C_0 f u$$

on $M \times [0, T]$. Assume that there hold on $M \times [0, T]$

$$u \leq c(n)f, \quad \frac{\partial}{\partial t} dv_t \leq c(n)f dv_t.$$

Define

$$e_0(t) = \sup_{x \in M, 0 \leq \tau \leq t} \left(\int_{B_{r/2}(x)} f^{\frac{n}{2}} dv_\tau \right)^{2/n}. \quad (2.13)$$

Then we have the following proposition of f and u .

Proposition 2.3. *Let f and u be as above, A be given by (2.1) and $e_0(t)$ be defined by (2.13). Suppose there holds for all $x \in M$*

$$\left(\int_{B_{r/2}(x)} f_0^{\frac{n}{2}} dv_0 \right)^{\frac{2}{n}} \leq (2N^{1+\frac{2}{n}} n(C_0 + c(n))A)^{-1},$$

where $N = N(n, K)$ is given by (2.12), $f_0(x) = f(x, 0)$ and $dv_0 = dv_{g_0}$. Then there exist two constants C_1 and C_2 depending only on n and C_0 such that if $0 < t < \min(T, C_2 N^{-1} r^2)$, then $f(x, t) \leq C_1 t^{-1}$ and

$$u(x, t) \leq C_1 A^{\frac{n}{n+2}} t^{-\frac{n}{n+2}} \left[\left(\int_{B_r(x)} u_0^{\frac{n+2}{2}} dv_0 \right)^{\frac{2}{n+2}} + r^{-\frac{4}{n+2}} e_0(t) \right].$$

Proof. Let $[0, T'] \subset [0, T]$ be the maximal interval such that

$$e_0(T') = \sup_{x \in M, 0 \leq t \leq T'} \left(\int_{B_{r/2}(x)} f^{\frac{n}{2}} dv_t \right)^{\frac{2}{n}} \leq ((C_0 + c(n))nNA)^{-1}. \quad (2.14)$$

For any cut-off function ϕ supported in $B_r(x)$, using the same method of deriving (2.7), we

calculate when $p \leq n$ and $m \leq n$,

$$\begin{aligned}
\frac{1}{p} \frac{\partial}{\partial t} \int \phi^{m+2} f^p dv_t &\leq \int \phi^{m+2} f^{p-1} (\Delta f + C_0 f^2) dv_t + \frac{c(n)}{p} \int \phi^{m+2} f^{p+1} dv_t \\
&\leq - \int \nabla(\phi^{m+2} f^{p-1}) \nabla f dv_t + \left(C_0 + \frac{c(n)}{p} \right) \\
&\quad \times \left(\int_{B_{2r}(x)} f^{\frac{n}{2}} dv_t \right)^{\frac{2}{n}} \left(\int (\phi^{m+2} f^p)^{\frac{n}{n-2}} dv_t \right)^{\frac{n-2}{n}} \\
&\leq -\frac{2}{p} \int |\nabla(\phi^{\frac{m}{2}+1} f^{\frac{p}{2}})|^2 dv_t + \frac{2}{p} \int |\nabla \phi^{\frac{m}{2}+1}|^2 f^p dv_t \\
&\quad + \left(C_0 + \frac{c(n)}{p} \right) N e_0 A \int |\nabla(\phi^{\frac{m}{2}+1} f^{\frac{p}{2}})|^2 dv_t. \\
&\leq -\frac{1}{p} \int |\nabla(\phi^{\frac{m}{2}+1} f^{\frac{p}{2}})|^2 dv_t + \frac{(m+2)^2}{2p} |\nabla \phi|_{\infty}^2 \int \phi^m f^p dv_t.
\end{aligned}$$

Here in the second and third inequalities we used (2.12) and the Sobolev inequality. Hence

$$\frac{\partial}{\partial t} \int \phi^{m+2} f^p dv_t + \int |\nabla(\phi^{\frac{m}{2}+1} f^{\frac{p}{2}})|^2 dv_t \leq \frac{(m+2)^2}{2} |\nabla \phi|_{\infty}^2 \int \phi^m f^p dv_t. \quad (2.15)$$

Take ϕ supported in $B_r(x)$, which is 1 on $B_{r/2}(x)$ and $|\nabla_{g_0} \phi|_{\infty}^2 \leq 5/r^2$. Since $\frac{1}{2} g_{ij}(0) \leq g_{ij}(t) \leq 2g_{ij}(0)$, we have $|\nabla_{g(t)} \phi|_{\infty}^2 \leq 10/r^2$. Taking $p = \frac{n}{2}$ in (2.15) and integrating it from 0 to t , we obtain by using (2.12) again

$$\begin{aligned}
\int_{B_{r/2}(x)} f^{\frac{n}{2}} dv_t &\leq \int_{B_r(x)} f_0^{\frac{n}{2}} dv_0 + \frac{2(m+2)^2}{r^2} \int_0^t \int \phi^m f^{\frac{n}{2}} dv_t dt \\
&\leq N \left(2N^{1+\frac{2}{n}} n(C_0 + c(n))A \right)^{-\frac{n}{2}} + 2(m+2)^2 r^{-2} N(e_0(t))^{\frac{n}{2}} t. \quad (2.16)
\end{aligned}$$

Noting that x is arbitrary, one concludes

$$(1 - 2(m+2)^2 r^{-2} N t) (e_0(t))^{\frac{n}{2}} \leq N \left(2N^{1+\frac{2}{n}} n(C_0 + c(n))A \right)^{-\frac{n}{2}}.$$

If $T' < \frac{r^2}{8(m+2)^2 N}$, then for all $t \in [0, T']$

$$e_0(t) < \left(\frac{4}{3} \right)^{2/n} (2Nn(C_0 + c(n))A)^{-1}.$$

This contradicts the maximality of $[0, T']$. We can therefore assume that $T' \geq \min(C_2 N^{-1} r^2, T)$.

It follows from (2.15) that

$$\begin{aligned}
\frac{\partial}{\partial t} \left(t \int \phi^{m+2} f^p dv_t \right) &= t \frac{\partial}{\partial t} \int \phi^{m+2} f^p dv_t + \int \phi^{m+2} f^p dv_t \\
&\leq \left(\frac{(m+2)^2}{2} |\nabla \phi|_{\infty}^2 t + 1 \right) \int \phi^m f^p dv_t.
\end{aligned}$$

When $0 \leq t \leq \min(C_2 N^{-1} r^2, T)$, integrating the above inequality from 0 to t , we have

$$\begin{aligned} \int \phi^{m+2} f^p dv_t &\leq \left(\frac{2(m+2)^2}{r^2} + \frac{1}{t} \right) \int_0^t \int \phi^m f^p dv_t dt \\ &\leq c t^{-1} \int_0^t \int \phi^m f^p dv_t dt \end{aligned} \quad (2.17)$$

for some constant c depending only on n . Moreover, integrating (2.15) from 0 to t , we derive

$$\int_0^t \int |\nabla(\phi^{\frac{m}{2}+1} f^{\frac{p}{2}})|^2 dv_t dt \leq \int \phi^{m+2} f_0^p dv_0 + \frac{2(m+2)^2}{r^2} \int_0^t \int \phi^m f^p dv_t dt. \quad (2.18)$$

Noting that $\frac{1}{r^2} \leq \frac{C_2}{Nt}$ and $m \leq n$, we calculate by using (2.17) and (2.18)

$$\begin{aligned} \int_{B_{r/2}(x)} f^{\frac{n}{2}+1} dv_t &\leq \int_{B_r(x)} \phi^{m+4} f^{\frac{n}{2}+1} dv_t \\ &\leq C t^{-1} \int_0^t \int \phi^{m+2} f^{\frac{n}{2}+1} dv_t dt \\ &\leq C t^{-1} \int_0^t \left(\int_{B_r(x)} f^{\frac{n}{2}} dv_t \right)^{\frac{2}{n}} \left(\int (\phi^{m+2} f^{\frac{n}{2}})^{\frac{n}{n-2}} dv_t \right)^{\frac{n-2}{n}} dt \\ &\leq C t^{-1} N^{\frac{2}{n}} e_0(t) A \int_0^t \int |\nabla(\phi^{\frac{m}{2}+1} f^{\frac{n}{4}})|^2 dv_t dt \\ &\leq C t^{-1} N^{\frac{2}{n}} e_0(t) A (N(e_0(t))^{\frac{n}{2}} + N(e_0(t))^{\frac{n}{2}} t) \\ &\leq C N^{1+\frac{2}{n}} A (e_0(t))^{1+\frac{n}{2}} t^{-1}, \end{aligned}$$

or equivalently

$$\left(\int_{B_{r/2}(x)} f^{\frac{n+2}{2}} dv_t \right)^{\frac{2}{n+2}} \leq C N A^{\frac{2}{n+2}} e_0(t) t^{-\frac{2}{n+2}}, \quad (2.19)$$

where C is a constant depending only on n , here and in the sequel, we often denote various constants by the same C . Setting $q = n + 2$, $p = \frac{n}{2}$ and $\mu = C N A^{\frac{2}{n+2}} e_0(T')$, we obtain by employing Theorem 2.1

$$\begin{aligned} f(x, t) &\leq C A \left(\frac{1 + A^{\frac{n}{2}} \mu^{\frac{n+2}{2}}}{t} + \frac{1}{r^2} \right)^{\frac{n+2}{n}} \left(\int_0^t \int_{B_r(x)} f^{\frac{n}{2}} dv_t dt \right)^{\frac{2}{n}} \\ &\leq C A e_0(T') t^{\frac{2}{n}} \left(\frac{1 + A^{\frac{n}{2}} \mu^{\frac{n+2}{2}}}{t} + \frac{1}{r^2} \right)^{\frac{n+2}{n}} \end{aligned}$$

for $t \in [0, T']$. Recalling the definition of $e_0(T')$ (see (2.14) above), we can see that $A e_0(T')$ is bounded and

$$A^{\frac{n}{2}} \mu^{\frac{n+2}{2}} = (C N A e_0(T'))^{\frac{n+2}{2}} \quad (2.20)$$

is also bounded. Therefore when $0 < t < \min(T, C_2 N^{-1} r^2)$, $f(x, t) \leq C_1 t^{-1}$ for some constants C_1 and C_2 depending only on n, C_0 .

Using $u \leq c(n)f$ and $\partial_t dv_t \leq c(n)f dv_t$ and mimicking the method of proving (2.15), we obtain

$$\frac{\partial}{\partial t} \int \phi^{m+2} u^p dv_t + \int |\nabla(\phi^{\frac{m}{2}+1} u^{\frac{n}{2}})|^2 dv_t \leq \frac{C}{r^2} \int \phi^m u^p dv_t. \quad (2.21)$$

Taking $m = 0$, $p = n/2$ and integrating this inequality, we have by using (2.12))

$$\int_0^t \int |\nabla(\phi u^{\frac{n}{4}})|^2 dv_t dt \leq \int_{B_r(x)} u_0^{\frac{n}{2}} dv_0 + \frac{C}{r^2} N(e_0(t))^{\frac{n}{2}} t. \quad (2.22)$$

Integrating (2.21) with $m = 2$, $p = (n+2)/2$, and using the Sobolev inequality (2.1), we obtain

$$\begin{aligned} \int_{B_{r/2}(x)} u^{\frac{n+2}{2}} dv_t &\leq \int_{B_r(x)} u_0^{\frac{n+2}{2}} dv_0 + \frac{C}{r^2} \int_0^t \int \phi^2 u^{\frac{n+2}{2}} dv_t dt \\ &\leq \int_{B_r(x)} u_0^{\frac{n+2}{2}} dv_0 + \frac{C}{r^2} e_0(t) A \int_0^t \int |\nabla(\phi u^{\frac{n}{4}})|^2 dv_t dt, \end{aligned}$$

which together with (2.22) and (2.12) gives

$$\begin{aligned} \int_{B_{r/2}(x)} u^{\frac{n+2}{2}} dv_t &\leq \int_{B_r(x)} u_0^{\frac{n+2}{2}} dv_0 + \frac{C}{r^2} e_0(t) A \left(\int_{B_r(x)} u_0^{\frac{n}{2}} dv_0 + \frac{C}{r^2} N e_0(t)^{\frac{n}{2}} t \right) \\ &\leq \int_{B_r(x)} u_0^{\frac{n+2}{2}} dv_0 + \frac{C}{r^2} N A (e_0(t))^{1+\frac{n}{2}} \left(1 + \frac{1}{r^2} t \right). \end{aligned} \quad (2.23)$$

Notice that when $0 \leq t \leq \min(C_2 r^2/N, T)$, (2.19) implies

$$\int_{B_{r/2}(x)} f^{\frac{n+2}{2}} dv_t \leq \mu t^{-1}.$$

Without loss of generality we can assume $A > 1$ (otherwise we can substitute A for $A+1$). In view of (2.20) and (2.23), we obtain by using Theorem 2.1 in the case $q = n+2$ and $p = (n+2)/2$

$$\begin{aligned} u(x, t) &\leq C A^{\frac{n}{n+2}} \left(\frac{1}{t} + \frac{1}{r^2} \right) \left(\int_0^t \int_{B_{r/2}(x)} u^{\frac{n+2}{2}} dv_t dt \right)^{\frac{2}{n+2}} \\ &\leq C A^{\frac{n}{n+2}} t^{-\frac{n}{n+2}} \left[\left(\int_{B_r(x)} u_0^{\frac{n+2}{2}} dv_0 \right)^{\frac{2}{n+2}} + r^{-\frac{4}{n+2}} e_0(t) \right], \end{aligned}$$

provided that $0 \leq t \leq \min(C_2 r^2/N, T)$. □

Remark 2.4. We remark that Theorem 2.1 and Proposition 2.3 are very similar to Theorem A.1 and Corollary A.10 of Dean Yang's paper [17] respectively. The differences are that we have heat flow type inequalities, but Dean Yang has heat flow type inequalities with cut-off function. It seems that Dean Yang's Corollary A.10 is stronger than our Proposition 2.3, which is enough for our use here. Also we should compare Theorem 2.1 with ([6, 7], Theorem 2.1), where Dai-Wei-Ye obtained a similar result by using a similar method. Here the constant C of (2.6) depends only on n, q, p , but not on the Sobolev constant A . While in [6, 7], since the Sobolev constants $C_S(t)$ along the flow are bounded, they need not care how the constant C exactly depends on C_S .

3. Short time existence of the Ricci flow

In this section we focus on closed Riemannian manifolds. Precisely, following the lines of [13, 18], we study the short time existence of the Ricci flow and give the proof of Theorem 1.1. Assume (M, g_0) is a closed Riemannian manifold of dimension $n(\geq 3)$ with $|\text{Ric}(g_0)| \leq K$. Consider the Ricci flow

$$\begin{cases} \frac{\partial g}{\partial t} = -2\text{Ric}(g), \\ g(0) = g_0. \end{cases} \quad (3.1)$$

It is well known [10] that the Riemannian curvature tensor and the Ricci curvature tensor satisfy the following evolution equations

$$\frac{\partial \text{Rm}}{\partial t} = \Delta \text{Rm} + \text{Rm} * \text{Rm}, \quad (3.2)$$

$$\frac{\partial \text{Ric}}{\partial t} = \Delta \text{Ric} + \text{Rm} * \text{Ric}, \quad (3.3)$$

where $\text{Rm} * \text{Rm}$ is a tensor that is quadratic in Rm , $\text{Ric} * \text{Ric}$ can be understood in a similar way. It follows that

$$\frac{\partial |\text{Rm}|}{\partial t} \leq \Delta |\text{Rm}| + c(n)|\text{Rm}|^2, \quad (3.4)$$

$$\frac{\partial |\text{Ric}|}{\partial t} \leq \Delta |\text{Ric}| + c(n)|\text{Rm}||\text{Ric}|. \quad (3.5)$$

To prove Theorem 1.1, it suffices to prove the following:

Proposition 3.1. *Let (M, g_0) be a closed Riemannian manifold of dimension $n(\geq 3)$ with $|\text{Ric}(g_0)| \leq K$. Suppose there exists a constant $A_0 > 0$ such that the following local Sobolev inequalities hold for all $x \in M$*

$$\|u\|_{2n/(n-2)}^2 \leq A_0 \|\nabla u\|_2^2, \quad \forall u \in C_0^\infty(B_r(x)).$$

Then there exist constants C_1, C_3 depending only on n and K , and C_2 depending only on n such that for $r \leq 1$, if

$$\left(\int_{B_{r/2}(x)} |\text{Rm}(g_0)|^{\frac{n}{2}} dv_{g_0} \right)^{2/n} \leq (C_1 A_0)^{-1}$$

for all $x \in M$, then the Ricci flow (3.1) has a smooth solution for $0 \leq t \leq T$, where $T \geq C_2 \min(r^2/N, K^{-1})$, such that for all $x \in M$

$$\frac{1}{2}g_0 \leq g(t) \leq 2g_0, \quad (3.6)$$

$$\|u\|_{2n/(n-2)}^2 \leq 4A_0 \|\nabla u\|_2^2, \quad \forall u \in C_0^\infty(B_r(x)), \quad (3.7)$$

$$\left(\int_{B_{r/2}(x)} |\text{Rm}(g(t))|^{\frac{n}{2}} dv_t \right)^{2/n} \leq 2N(C_1 A_0)^{-1}. \quad (3.8)$$

Proof. It is well known (see for example [8, 10]) that a smooth solution $g(t)$ of the Ricci flow (3.1) exists for a short time interval and is unique. Let $[0, T_{\max})$ be a maximum time interval on which $g(t)$ exists and (3.6)-(3.8) hold. Clearly $T_{\max} > 0$ since the strict inequalities in (3.6)-(3.8)

hold at $t = 0$. Suppose $T_{\max} < T_0 = C_2 \min(r^2/N, K^{-1})$ for some constant C_2 to be determined later. Since the Ricci curvature satisfies (3.5), it follows from Proposition 2.3 that for $0 \leq t \leq T'$,

$$\begin{aligned} |\text{Ric}(g(t))| &\leq CA_0^{\frac{n}{n+2}} t^{-\frac{n}{n+2}} \left[\left(\int_{B_r(x)} |\text{Ric}(g_0)|^{\frac{n+2}{2}} dv_0 \right)^{\frac{2}{n+2}} + r^{-\frac{4}{n+2}} e_0(T') \right] \\ &\leq CA_0^{\frac{n}{n+2}} t^{-\frac{n}{n+2}} \left(K^{\frac{2}{n+2}} (e_0(T'))^{\frac{n}{n+2}} + r^{-\frac{4}{n+2}} e_0(T') \right) \\ &\leq C(K^{\frac{2}{n+2}} + r^{-\frac{4}{n+2}}) t^{-\frac{n}{n+2}}, \end{aligned} \quad (3.9)$$

where T' and $e_0(T')$ are defined by (2.14) in the case f is replaced by $|\text{Rm}|$. It follows that for all $x \in M$, $u \in C_0^\infty(B_r(x))$ and $0 \leq t \leq T'$,

$$\begin{aligned} \left| \frac{d}{dt} \int_{B_r(x)} |u|^{\frac{2n}{n-2}} dv_t \right| &\leq 2|\text{Ric}(g(t))|_\infty \int_{B_r(x)} |u|^{\frac{2n}{n-2}} dv_t \\ &\leq C t^{-\frac{n}{n+2}} \int_{B_r(x)} |u|^{\frac{2n}{n-2}} dv_t. \end{aligned}$$

This implies

$$e^{-C t^{\frac{2}{n+2}}} \int_{B_r(x)} |u|^{\frac{2n}{n-2}} dv_0 \leq \int_{B_r(x)} |u|^{\frac{2n}{n-2}} dv_t \leq e^{C t^{\frac{2}{n+2}}} \int_{B_r(x)} |u|^{\frac{2n}{n-2}} dv_0.$$

Similarly we have

$$\left| \frac{d}{dt} \int_{B_r(x)} |\nabla u|^2 dv_t \right| \leq C t^{-\frac{n}{n+2}} \int_{B_r(x)} |\nabla u|^2 dv_t,$$

and

$$e^{-C t^{\frac{2}{n+2}}} \int_{B_r(x)} |\nabla u|^2 dv_0 \leq \int_{B_r(x)} |\nabla u|^2 dv_t \leq e^{C t^{\frac{2}{n+2}}} \int_{B_r(x)} |\nabla u|^2 dv_0.$$

Hence if $T_{\max} < T_0 = C_2 \min(r^2/N, K^{-1})$ for sufficiently small C_2 depending only on n and K , then (3.7) holds with strict inequality.

To show (3.6) holds with strict inequality, we fix a tangent vector v and calculate

$$\frac{d}{dt} |v|_{g(t)}^2 = \frac{d}{dt} (g_{ij}(t) v^i v^j) = -2\text{Ric}_{ij} v^i v^j,$$

which together with (3.9) gives

$$\left| \frac{d}{dt} \log |v|_{g(t)}^2 \right| \leq C(K^{\frac{2}{n+2}} + r^{-\frac{4}{n+2}}) t^{-\frac{n}{n+2}}.$$

Therefore we obtain for $0 \leq t < C_2 \min(r^2, K^{-1})$,

$$\frac{1}{2} |v|_{g(0)}^2 < |v|_{g(t)}^2 < 2 |v|_{g(0)}^2.$$

Using the same method of deriving (2.16), one can see that the strict inequality in (3.8) holds when $0 \leq t < C_2 \min(r^2, K^{-1})$ for sufficiently small C_2 . By Proposition 2.3, $|\text{Rm}(g(t))|_\infty \leq C t^{-1}$ for all $t \in [0, T_{\max}]$. Hence one can extend $g(t)$ smoothly beyond T_{\max} with (3.6)-(3.8) still holding. This contradicts the assumed maximality of T_{\max} . Therefore $T_{\max} \geq T_0$. \square

Proof of Theorem 1.1. By Proposition 3.1, there exists a unique solution $g(t)$ of the Ricci flow (3.1) such that (3.6)-(3.8) hold. Then by Proposition 2.3, one concludes

$$|\text{Rm}(g(t))| \leq Ct^{-1}, \quad |\text{Ric}(g(t))| \leq Ct^{-\frac{n}{n+2}}$$

for $t \in [0, T_0]$. This completes the proof of Theorem 1.1. \square

4. Applications

In this section, we will prove Theorem 1.2 by applying Theorem 1.1. It follows from (1.4)-(1.6) that the deformed metric $g(t)$ has uniform sectional curvature bounds away from $t = 0$ and $g(t)$ is close to $g(0)$ when t is close to 0. We first show that diameters of the flow are under control, namely

Lemma 4.1. *Let $g(t)$ be the Ricci flow in Theorem 1.1. Then for $0 \leq t \leq c_1 \min(r^2, K^{-1})$, there exists a constant c depending only on n and K such that*

$$e^{-ct^{\frac{2}{n+2}}} \text{diam}(g_0) \leq \text{diam}(g(t)) \leq e^{ct^{\frac{2}{n+2}}} \text{diam}(g_0). \quad (4.1)$$

where $\text{diam}(g(t))$ means the diameter of the manifold $(M, g(t))$.

Proof. Let $\gamma : [0, 1] \rightarrow M$ be any smooth curve. Denote the length of γ by

$$l_\gamma(t) = \int_0^1 |\dot{\gamma}(s)|_{g(t)}^2 ds.$$

We calculate by using the Ricci bound in Theorem 1.2

$$\left| \frac{d}{dt} l_\gamma(t) \right| = \left| \int_0^1 -2\text{Ric}_{g(t)}(\dot{\gamma}(s), \dot{\gamma}(s)) ds \right| \leq ct^{-\frac{n}{n+2}} l_\gamma(t).$$

This implies

$$l_\gamma(0)e^{-ct^{\frac{2}{n+2}}} \leq l_\gamma(t) \leq l_\gamma(0)e^{ct^{\frac{2}{n+2}}}.$$

It follows that

$$e^{-ct^{\frac{2}{n+2}}} \text{dist}_{g_0}(p, q) \leq \text{dist}_{g(t)}(p, q) \leq e^{ct^{\frac{2}{n+2}}} \text{dist}_{g_0}(p, q),$$

where $\text{dist}_{g(t)}(p, q)$ denote the distance between p and q in the metric $g(t)$. This gives the desired result. \square

The following proposition is a corollary of Gromov's almost flat manifold theorem [9]:

Proposition 4.2 (Gromov). *Let (M, g) be a compact Riemannian manifold of dimension n . Assume the sectional curvature is bounded, i.e., $|\text{Sec}(g)| \leq \Lambda$. Then there exists a constant ϵ_0 depending only on n such that if*

$$\Lambda(\text{diam}(g))^2 \leq \epsilon_0, \quad (4.2)$$

then the universal covering of (M, g) is diffeomorphic to \mathbb{R}^n . If in addition the fundamental group $\pi(M)$ is commutative, then (M, g) is diffeomorphic to a torus.

Proof of Theorem 1.2. Let $g(t)$ be a unique solution to the Ricci flow (1.3). By (1.5), for $0 \leq t \leq c_1 \min(r^2, K^{-1})$,

$$|\text{Sec}(g(t))| \leq ct^{-1},$$

where $\text{Sec}(g(t))$ denotes the sectional curvature of $(M, g(t))$. Let ϵ_0 be given by Proposition 4.2. Take $t_0 = c_1 \min(r^2, K^{-1})$ and

$$\delta = \left(\epsilon_0 t_0 c^{-1} e^{-2ct_0^{\frac{2}{n+2}}} \right)^{1/2}.$$

If $\text{diam}(g_0) \leq \delta$, then we obtain by Lemma 4.1

$$|\text{Sec}(g(t_0))(\text{diam}(g(t_0)))^2| \leq ct_0^{-1} e^{2ct_0^{\frac{2}{n+2}}} (\text{diam}(g_0))^2 \leq \epsilon_0.$$

Applying Proposition 4.2 to $g(t_0)$, we conclude Theorem 1.2. \square

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